

# A NOTE ON THE NON-EXISTENCE OF SMALL COHEN-MACAULAY ALGEBRAS

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**ABSTRACT.** By finding a  $p$ -adic obstruction, we construct many examples of complete noetherian local normal  $\mathbf{F}_p$ -algebras  $R$  such that no module-finite extension  $R \hookrightarrow S$  is Cohen-Macaulay. These examples should be contrasted with a result of Hochster-Huneke: the directed union of all such extensions is always Cohen-Macaulay.

## 1. INTRODUCTION

A noetherian ring is called Cohen-Macaulay (CM) if there are no non-trivial relations between elements of a system of parameters. These rings form an exceptionally well behaved class: local cohomology vanishes when it can, and duality works out beautifully. This note studies rings realisable as subrings of CM rings:

**Definition 1.1.** A noetherian ring  $R$  admits a *big CM algebra* if there is an injective map  $R \hookrightarrow S$  of rings with  $S$  CM. If the ring  $S$  can be chosen to be finitely generated as an  $R$ -module, then we say that  $R$  admits a *small CM algebra*.

Under mild hypotheses, a fundamental result of Hochster and Huneke [HH92] shows that any ring  $R$  that contains a field admits a big CM algebra; in fact, if  $R$  is an  $\mathbf{F}_p$ -algebra, the absolute integral closure of  $R$  does the job. Thus, one asks: does a ring  $R$  always admit a small CM algebra? The answer is “no” in characteristic 0 based on a local cohomological obstruction<sup>1</sup>. In characteristic  $p$ , however, the results of *loc. cit.* immediately nullify any *coherent* cohomological obstructions: any non-trivial relation can be trivialised after a finite extension. In fact, by [HL07], there is a single finite extension  $R \hookrightarrow S$  trivialising *all* unwanted relations. Nevertheless, our goal in this note is to discuss a negative answer to the above question in characteristic  $p$  using a  $p$ -adic (rigid) cohomological obstruction.

**Theorem 1.2.** Let  $(A, L)$  be polarised projective variety over a perfect field  $k$ ; set  $\widehat{R}$  to be completion at the origin of  $\bigoplus_{n \geq 0} H^0(A, L^n)$ . Assume  $H_{\text{rig}}^i(A)_{<1} \neq 0$  for some  $0 < i < \dim(A)$ . Then  $\widehat{R}$  does not admit a small CM algebra.

The hypothesis on  $A$  is satisfied, for example, if  $A$  is an abelian surface. This theorem generalises a calculation of Sannai-Singh [SS12, Example 5.3] (who showed the non-existence of a *graded* small CM algebra for a specific  $A$ ). This theorem should also be contrasted with a result of Hartshorne: if  $A$  is CM, then  $R$  admits a small CM *module*, i.e., a module whose depth is  $\dim(A)$  (see [Hoc75]). We end by noting that the proof of Theorem 1.2 shows a stronger statement: any local  $\mathbf{F}_p$ -algebra that admits a small CM algebra is “Witt Cohen-Macaulay,” see Remark 2.12.

**A summary of the proof.** Let us informally explain the proof in an example:  $A$  is an ordinary abelian surface. The key idea is to work  $p$ -adically instead of modulo  $p$  until the end to track the divisibility properties of local cohomology under finite extensions. More precisely, the  $p$ -rank of  $A$  contributes free  $\mathbf{Z}_p$ -summands to  $H_{\text{m}}^2(\text{Spec}(\widehat{R})_{\text{ét}}, \mathbf{Z}_p)$ . Trace formalism then shows: for any finite extension  $\widehat{R} \rightarrow \widehat{S}$ , the group  $H_{\text{m}}^2(\text{Spec}(\widehat{S})_{\text{ét}}, \mathbf{Z}_p)$  also contains free  $\mathbf{Z}_p$ -summands. Reducing these modulo  $p$  and using the Artin-Schreier sequence shows  $H_{\text{m}}^2(\widehat{S}) \neq 0$ , so  $\widehat{S}$  is not CM. In the body of the note, we work with Witt-vector cohomology instead of  $p$ -adic étale cohomology to apply the preceding argument with fewer ordinarity constraints, see Remark 2.2.

**Notation.** For any  $\mathbf{F}_p$ -scheme  $X$ , we write  $\{W_n \mathcal{O}_X\}$  of the projective system of (Zariski) sheaves defined by the truncated Witt vector functors (see [Ill79, Ser58]), and  $W \mathcal{O}_X := \varprojlim_n W_n \mathcal{O}_X$  for its inverse limit (which is also the derived limit: the transition maps  $W_n \mathcal{O}_X \rightarrow W_{n-1} \mathcal{O}_X$  are surjective for all  $n$ , and  $W_n \mathcal{O}_X$  has no higher cohomology on affines). For a closed subset  $Z \subset X$  and any abelian sheaf  $A$ , we write  $\text{R}\Gamma_Z(X, A)$  for the homotopy-kernel of  $\text{R}\Gamma(X, A) \rightarrow \text{R}\Gamma(X - Z, A)$ . In particular, since cohomology commutes with derived limits, we have  $\text{R}\Gamma_Z(X, W \mathcal{O}_X) \simeq \text{R}\varprojlim_n \text{R}\Gamma_Z(X, W_n \mathcal{O}_X)$ . Recall that  $\text{R}\Gamma(X, -)$  commutes with filtered colimits of sheaves if  $X$  is quasi-compact and quasi-separated. In particular, if both  $X$  and  $X - Z$  are quasi-compact and quasi-separated, then  $\text{R}\Gamma_Z(X, W \mathcal{O}_X, \mathbf{Q}) \simeq (\text{R}\varprojlim_n \text{R}\Gamma_Z(X, W_n \mathcal{O}_X)) \otimes_{\mathbf{Z}} \mathbf{Q}$ .

<sup>1</sup>A normal local noetherian  $\mathbf{Q}$ -algebra  $R$  that admits a small CM algebra  $S$  is itself CM:  $H_{\text{m}}^i(R)$  is a summand of  $H_{\text{m}}^i(S)$  via the trace splitting.

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## 2. PROOF

**2.1. Remarks on Witt vectors.** We use  $k$  to denote a fixed perfect base field of characteristic  $p$ . We start by recalling a fundamental result identifying the slope  $< 1$  part of rigid cohomology with Witt vector cohomology; the smooth case is due to Bloch-Deligne-Illusie (see [Blo77, Theorem 0.2] and [Ill79, §II.3]), while the singular case is newer:

**Theorem 2.1** ([BBE07, Theorem 1.1]). *Let  $A$  be a proper  $k$ -variety. Then  $H^i(A, W\mathcal{O}_A)_{\mathbf{Q}} \simeq H_{\text{rig}}^i(A)_{<1}$ .*

**Remark 2.2.** Theorem 2.1 is the main reason we work with Witt-vector cohomology instead of  $p$ -adic étale cohomology in this note: the latter only describes the slope 0 part of rigid cohomology, while the former describes the (potentially much larger) slope  $< 1$  part. In particular, if  $A$  is an abelian variety, then  $H^1(A, W\mathcal{O}_{A,\mathbf{Q}})$  is always non-zero for weight reasons, while  $H^1(A, \mathbf{Z}_p)$  vanishes if  $A$  is supersingular and  $k = \bar{k}$ .

The next lemma allows deduction of modulo  $p$  consequences from rational assumptions.

**Lemma 2.3.** *Let  $Y$  be a  $k$ -scheme with a closed subscheme  $Z$ . If  $H_Z^i(Y, W\mathcal{O}_Y)_{\mathbf{Q}} \neq 0$  for some  $i$ , then  $H_Z^j(Y, \mathcal{O}_Y) \neq 0$  for  $j = i$  or  $j = i - 1$ .*

*Proof.* The assumption on  $H_Z^i(Y, W\mathcal{O}_Y)_{\mathbf{Q}}$  and the formula

$$\mathrm{R}\Gamma_Z(Y, W\mathcal{O}_Y)_{\mathbf{Q}} \simeq (\mathrm{R}\lim_n \mathrm{R}\Gamma_Z(Y, W_n\mathcal{O}_Y)) \otimes_{\mathbf{Z}} \mathbf{Q}$$

show that  $H_Z^j(Y, W_n\mathcal{O}_Y) \neq 0$  for some  $n > 0$  and some  $j \in \{i - 1, i\}$  (as  $\mathrm{R}^i \lim = 0$  for  $i > 1$ ). The rest follows by standard exact sequences expressing  $W_n\mathcal{O}_Y$  as an iterated extension of copies of  $\mathcal{O}_Y$ .  $\square$

We need trace maps in Witt vector cohomology, so we recall a direct construction (essentially due to [SV96]).

**Lemma 2.4.** *Let  $f : Y \rightarrow X$  be a finite surjective morphism of noetherian normal schemes. For any abelian sheaf  $A$  on  $X_{\text{ét}}$  that is representable by an algebraic space, there is a functorial trace map  $\mathrm{Tr} : f_* f^* A \rightarrow A$  such that the composite  $A \xrightarrow{f^*} f_* f^* A \xrightarrow{\mathrm{Tr}} A$  is multiplication by the generic degree of  $f$ .*

*Proof.* We refer the reader to [Bha, Proposition 6.2] for a proof.  $\square$

**Remark 2.5.** The trace constructed in Lemma 2.4 is non-standard and slightly ad hoc. For example, if  $f : X \rightarrow X$  is the Frobenius map, then  $f^* : \mathrm{Shv}(X_{\text{ét}}) \rightarrow \mathrm{Shv}(X_{\text{ét}})$  is an equivalence, so the “correct” trace map should be an equivalence, while the one from Lemma 2.4 is multiplication by  $\deg(f)$  (composed with the inverse of  $\mathrm{id} \xrightarrow{\sim} f_* f^*$ ). In particular, the trace map constructed in Lemma 2.4 does *not* furnish a right adjoint to  $f_! \simeq f_*$ .

Using trace maps, we show that (rational) Witt vector cohomology cannot be killed by finite covers.

**Corollary 2.6.** *Let  $f : Y \rightarrow X$  be a finite surjective morphism of noetherian normal  $k$ -schemes, and let  $Z \subset X$  be a closed subset. Then  $H_Z^i(X, W\mathcal{O}_X)_{\mathbf{Q}} \rightarrow H_{f^{-1}Z}^i(Y, W\mathcal{O}_Y)_{\mathbf{Q}}$  is a direct summand.*

*Proof.* Let  $d$  be the generic degree of  $f$ . As  $W_n(-)$  is representable for each  $n > 0$ , Lemma 2.4 gives maps  $f_* W_n\mathcal{O}_Y \rightarrow W_n\mathcal{O}_X$  whose composition with the pullbacks  $W_n\mathcal{O}_X \rightarrow f_* W_n\mathcal{O}_Y$  is multiplication by  $d$ . These maps are compatible as  $n$  varies, so taking limits (and commuting them with  $f_*$ ) gives a map  $f_* W\mathcal{O}_Y \rightarrow W\mathcal{O}_X$  whose composition with  $W\mathcal{O}_X \rightarrow f_* W\mathcal{O}_Y$  is multiplication by  $d$ . The claim follows by applying  $H_Z^i(X, - \otimes_{\mathbf{Z}} \mathbf{Q})$ .  $\square$

**2.2. The main theorem.** To prove Theorem 1.2, we first establish some notation.

**Notation 2.7.** Fix a polarised projective variety  $(A, L)$  of a perfect characteristic  $p$  field  $k$ ; set  $R = \bigoplus_{n \geq 0} H^0(A, L^n)$  to be the section ring with  $\mathfrak{m} \subset R$  the homogeneous maximal ideal. Set  $X = \mathrm{Spec}(R)$ ,  $Z = \{\mathfrak{m}\} \subset X$  and  $U = X - Z$  with  $\pi : U \rightarrow A$  realising  $U$  as the total space of  $L^{-1}$  over  $A$ . Set  $\widehat{X} = \mathrm{Spec}(\widehat{R})$ ,  $\widehat{U} = \widehat{X} \times_X U$ , and abusively let  $Z \subset \widehat{X}$  denote the closed point.

Next, we record the expected relation between the cohomology of  $A$  and  $U$ :

**Lemma 2.8.** *The natural map  $W_n \mathcal{O}_A \rightarrow \pi_* W_n \mathcal{O}_U$  is a direct summand for all  $n$ . In particular,  $H^i(A, W_n \mathcal{O}_A) \rightarrow H^i(U, W_n \mathcal{O}_U)$  is a direct summand for all  $i$  and  $n$ .*

*Proof.* As  $\pi$  is a  $\mathbf{G}_m$ -torsor, the natural map  $\mathcal{O}_A \rightarrow \pi_* \mathcal{O}_U$  realises the source as the weight 0 eigenspace of the target. The rest follows by taking products and observing that  $W_n(R) \simeq R^n$  as sets functorially in  $R$ .  $\square$

We can now prove the main theorem:

*Proof of Theorem 1.2.* As  $W_n \mathcal{O}_X$  is an extension of  $\mathcal{O}_X$  by  $W_{n-1} \mathcal{O}_X$ , induction and the affineness of  $X$  show  $H^i(X, W_n \mathcal{O}_X) = 0$  for  $i > 0$ . Standard sequences then identify  $H^i(U, W_n \mathcal{O}_U) \simeq H_Z^{i+1}(X, W_n \mathcal{O}_X)$  for all  $i, n > 0$ . Since  $H^0(U, W_n \mathcal{O}_U) = W_n(H^0(U, \mathcal{O}_U))$ , the system  $\{H^0(U, W_n \mathcal{O}_U)\}$  has no  $\lim^1$  (as  $W_n(R) \rightarrow W_{n-1}(R)$  is surjective for any ring  $R$ ), so

$$H^i(U, W \mathcal{O}_U)_{\mathbf{Q}} \simeq H_Z^{i+1}(X, W \mathcal{O}_X)_{\mathbf{Q}} \simeq H_Z^{i+1}(\widehat{X}, W \mathcal{O}_{\widehat{X}})_{\mathbf{Q}}$$

for  $i > 0$ , where the last isomorphism comes from the excision identification  $\{R\Gamma_Z(X, W_n \mathcal{O}_X)\} \simeq \{R\Gamma_Z(\widehat{X}, W_n \mathcal{O}_{\widehat{X}})\}$  of projective systems. Theorem 2.1 and Lemma 2.8 then show  $H_{\text{rig}}^i(A)_{<1} \simeq H^i(A, W \mathcal{O}_A)_{\mathbf{Q}}$  is a direct summand of  $H_Z^{i+1}(\widehat{X}, W \mathcal{O}_{\widehat{X}})_{\mathbf{Q}}$  for  $i > 0$ . Choose  $0 < i < \dim(A)$  such that  $H_{\text{rig}}^i(A)_{<1} \neq 0$ . Let  $f : \widehat{Y} \rightarrow \widehat{X}$  be a finite surjective morphism of noetherian normal schemes; we will show that  $\widehat{Y}$  is not CM along  $f^{-1}Z$ . Corollary 2.6 shows  $H_{f^{-1}Z}^{i+1}(\widehat{Y}, W \mathcal{O}_{\widehat{Y}})_{\mathbf{Q}} \neq 0$ . Lemma 2.3 then gives  $H_{f^{-1}Z}^j(\widehat{Y}, \mathcal{O}_{\widehat{Y}}) \neq 0$  for some  $j \in \{i, i+1\}$ , which proves the claim as  $i+1 < \dim(A) + 1 = \dim(\widehat{Y})$ .  $\square$

**Remark 2.9.** As rigid cohomology is a Weil cohomology theory, the hypothesis on  $A$  in Theorem 1.2 may be reformulated topologically, at least when  $A$  is smooth and  $k$  is finite, to say: for some  $0 < i < \dim(A)$ , there is at least one Frobenius eigenvalue on  $H^i(A_{\bar{k}}, \mathbf{Q}_{\ell})$  which is not divisible by  $p$  (for some auxilliary prime  $\ell$  invertible on  $k$ ). Indeed, the eigenvalues occurring in  $H_{\text{rig}}^i(A)$  coincide with those on  $H^i(A_{\bar{k}}, \mathbf{Q}_{\ell})$  and are algebraic integers.

**Example 2.10.** Some elementary examples of projective varieties  $A$  to which Theorem 1.2 applies include: any projective variety of dimension  $\geq 2$  dominating a positive dimensional abelian variety, any projective variety of dimension  $\geq 3$  dominating a  $K3$  surface of finite height, etc.

We give an example showing that the presence of non-trivial middle cohomology of the structure sheaf does *not* force the non-existence of a small CM algebra for the section ring, even for smooth projective varieties; this example also shows the necessity of making a  $p$ -adic (rather than modulo  $p$ ) assumption on  $A$  in Theorem 1.2.

**Example 2.11.** Assume  $k$  is algebraically closed, and let  $X$  be a smooth complete intersection in some  $\mathbf{P}_k^n$  with  $\dim(X) \geq 2$  with the property that a suitable subgroup  $G \simeq \mathbf{F}_p \subset \text{GL}_{n+1}(k)$  preserves  $X$  and acts fixed point freely on  $X$ ; such examples were constructed by Serre, see [III05, §6]. Let  $Y = X/G$  denote the quotient. Then  $X \rightarrow Y$  is a finite étale  $G$ -torsor, and  $Y$  is a smooth projective  $k$ -variety; explicitly, the line bundle  $\mathcal{O}(1) \in \text{Pic}(X)$  is equivariant for the  $G$ -action, and hence descends to an ample line bundle  $L$  on  $Y$ . Let  $R$  and  $S$  denote the section rings of  $(Y, L)$  and  $(X, \mathcal{O}(1))$  respectively. Then  $R \rightarrow S$  is a finite extension. We will show that  $S$  is CM (even lci), but  $R$  is not. The former follows immediately as  $(X, \mathcal{O}(1))$  is a complete intersection. On the other hand, the Lefschetz formalism (see [Gro68, Corollary XII.3.5]) shows that  $X$  is simply connected. The Leray spectral sequence for  $X \rightarrow Y$  then yields  $\mathbf{F}_p \simeq G \simeq \pi_1^{\text{ét}}(Y) \simeq H^1(Y_{\text{ét}}, \mathbf{F}_p)^{\vee}$ , and hence  $H^1(Y, \mathcal{O}_Y) \neq 0$  by the Artin-Schreier sequence. The latter group is a direct summand of  $H_m^2(R)$ , so  $R$  is not CM (as  $\dim(R) = \dim(X) + 1 \geq 3$ ).

**Remark 2.12.** Let  $(R, \mathfrak{m})$  be a noetherian local  $k$ -algebra. The proof of Theorem 1.2 shows that being “Witt Cohen-Macaulay” (i.e., having  $H_m^i(\text{Spec}(R), W \mathcal{O}_{R, \mathbf{Q}}) = 0$  for  $i < \dim(R)$ ) is necessary for the existence of small CM algebras. We do not know if it is a sufficient condition. A (weakened) graded analogue asks: any projective variety  $X$  with  $H^i(X, W \mathcal{O}_{X, \mathbf{Q}}) = 0$  for  $0 < i < \dim(X)$  admits an alteration  $\pi : Y \rightarrow X$  with  $H^i(Y, \mathcal{O}_Y) = 0$  for  $0 < i < \dim(Y)$ . The simplest non-trivial instance of this question is when  $X = S \times \mathbf{P}^1$  with  $S$  a supersingular  $K3$  surface, where a positive answer is implied by Artin’s conjecture on unirationality of supersingular  $K3$  surfaces.

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